

X100/13/01

NATIONAL TUESDAY, 6 MAY
QUALIFICATIONS 1.00 PM – 4.00 PM
2014

MATHEMATICS
ADVANCED HIGHER

Read carefully

- 1 Calculators may be used in this paper.
- 2 Candidates should answer **all** questions.
- 3 **Full credit will be given only where the solution contains appropriate working.**



Answer all the questions

1. (a) Given

$$f(x) = \frac{x^2 - 1}{x^2 + 1},$$

obtain $f'(x)$ and simplify your answer.

3

- (b) Differentiate
- $y = \tan^{-1}(3x^2)$
- .

3

2. Write down and simplify the general term in the expression
- $\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}$
- .

Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$.

5

3. Use Gaussian elimination on the system of equations below to give an expression for
- z
- in terms of
- λ
- .

$$x + y + z = 2$$

$$4x + 3y - \lambda z = 4$$

$$5x + 6y + 8z = 11$$

For what values of λ does this system have a solution?

Determine the solution to this system of equations when $\lambda = 2$.

6

4. Given
- $x = \ln(1 + t^2)$
- ,
- $y = \ln(1 + 2t^2)$
- use parametric differentiation to find

$\frac{dy}{dx}$ in terms of t .

3

5. Three vectors
- \vec{OA}
- ,
- \vec{OB}
- and
- \vec{OC}
- are given by
- \mathbf{u}
- ,
- \mathbf{v}
- and
- \mathbf{w}
- where

$$\mathbf{u} = 5\mathbf{i} + 13\mathbf{j}, \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{w} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}.$$

Calculate $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

3

Interpret your result geometrically.

1

6. Given $e^y = x^3 \cos^2 x$, $x > 0$, show that

$$\frac{dy}{dx} = \frac{a}{x} + b \tan x, \text{ for some constants } a \text{ and } b.$$

State the values of a and b .

3

7. Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,

prove by induction that

$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, n \geq 1.$$

4

8. Find the solution $y = f(x)$ to the differential equation

$$4 \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = 0$$

given that $y = 4$ and $\frac{dy}{dx} = 3$ when $x = 0$.

6

9. Give the first three non-zero terms of the Maclaurin series for $\cos 3x$.

2

Write down the first four terms of the Maclaurin series for e^{2x} .

1

Hence, or otherwise, determine the Maclaurin series for $e^{2x} \cos 3x$

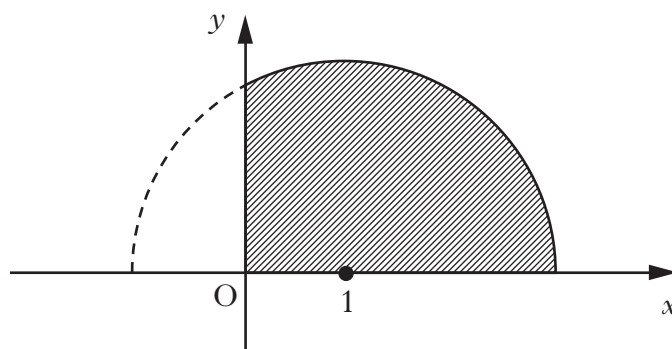
up to, and including, the term in x^3 .

3

10. A semi-circle with centre $(1, 0)$ and radius 2, lies on the x -axis as shown.

Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x -axis.

5

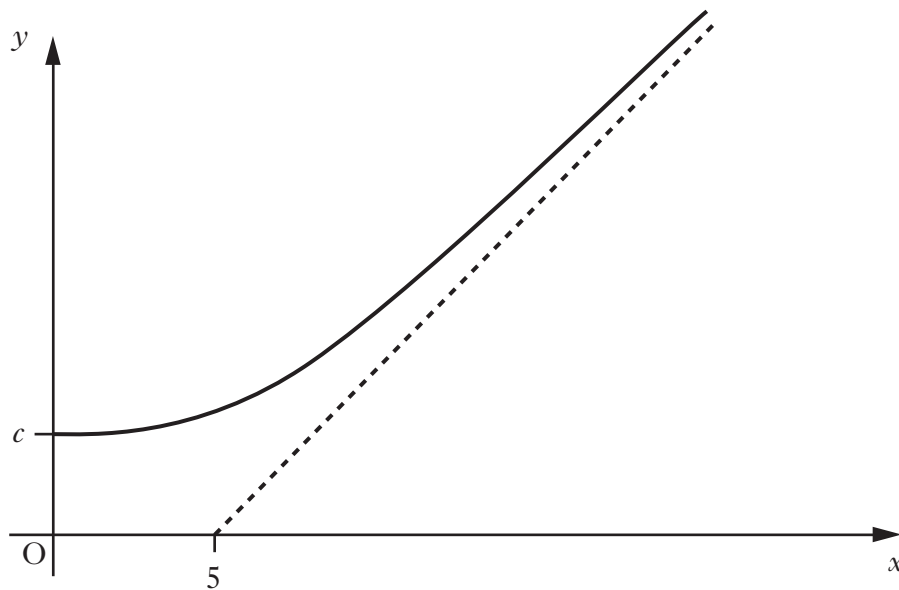


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11. The function $f(x)$ is defined for all $x \geq 0$.

The graph of $y = f(x)$ intersects the y -axis at $(0, c)$, where $0 < c < 5$.

The graph of the function and its asymptote, $y = x - 5$, are shown below.



(a) Copy the above diagram.

On the same diagram, sketch the graph of $y = f^{-1}(x)$.

Clearly show any points of intersection and any asymptotes.

4

(b) What is the equation of the asymptote of the graph of $y = f(x + 2)$?

1

(c) Why does your diagram show that the equation $x = f(f(x))$ has at least one solution?

1

12. Use the substitution $x = \tan \theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}}.$$

6

13. The fuel efficiency, F , in km per litre, of a vehicle varies with its speed, s km per hour, and for a particular vehicle the relationship is thought to be

$$F = 15 + e^x(\sin x - \cos x - \sqrt{2}), \quad \text{where } x = \frac{\pi(s-40)}{80},$$

for speeds in the range $40 \leq s \leq 120$ km per hour.

What is the greatest and least efficiency over the range and at what speeds do they occur?

10

14. (a) Given the series $1 + r + r^2 + r^3 + \dots$, write down the sum to infinity when $|r| < 1$.

Hence obtain an infinite geometric series for $\frac{1}{2-3r}$.

For what values of r is this series valid?

4

- (b) Express $\frac{1}{3r^2 - 5r + 2}$ in partial fractions.

Hence, or otherwise, determine the first three terms of an infinite series

for $\frac{1}{3r^2 - 5r + 2}$.

For what values of r does the series converge?

6

15. (a) Use integration by parts to obtain an expression for

$$\int e^x \cos x \, dx.$$

4

- (b) Similarly, given $I_n = \int e^x \cos nx \, dx$ where $n \neq 0$,

obtain an expression for I_n .

4

- (c) Hence evaluate $\int_0^{\frac{\pi}{2}} e^x \cos 8x \, dx$.

2

16. (a) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$.

3

- (b) Write down the four solutions to the equation $z^4 - 1 = 0$.

2

- (c) Plot the solutions of both equations on an Argand diagram.

1

- (d) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$.

2

- (e) Hence identify all the solutions to the equation

$$z^6 + z^4 + z^2 + 1 = 0.$$

2

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